Here g_e is the acceleration due to gravity at the earth's surface. Note that the choice of g_e is arbitrary. The advantage is that in all common systems (fps, cgs, SI etc.) the unit of specific impulse (I_{sp}) is the same '*seconds*'.

Thrust:

$$T = \dot{m}_e U_e + A_e(p_e - p_a) = \dot{m}_e U_{eq}$$

where $U_{eq} = U_e + \left(\frac{p_e - p_a}{\dot{m}}\right) A_e$

Maximum Thrust:

$$dT = \dot{m}_e dU_e + dA_e(p_e - p_a) + A_e dp_e = 0$$

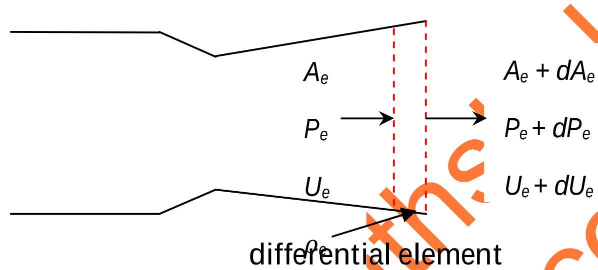

Figure 2 A differential element at the nozzle exit

momentum equation for a differential element (Fig.2) at exit gives

$$(\rho_e + d\rho_e)(U_e + dU_e)^2(A_e + dA_e) - \rho_e U_e^2 A_e = \sum F_x = A_e P_e - (A_e + dA_e)(P_e + dP_e) + P_e dA_e$$

$$\rho_e U_e A_e dU_e + A_e dp_e = \dot{m}_e dU_e + A_e dp_e = 0$$

$dT = dA_e(p_e - p_a) = 0$, or $p_e = p_a$

$$d^2T = d^2A_e(p_e - p_a) + dA_e dp_e$$

$$d^2T\Big|_{at\ p_e = p_a} = dA_e dp_e$$

Now $dA_e dp_e < 0$ as $dA_e > 0$ *and* $dp_e < 0$

Therefore, $p_e = p_a$ refers to optimum expansion. Under-expansion $p_e > p_a$ implies additional force is remaining unused and over-expansion $p_e < p_a$ can be noted as beginning of negative contribution to thrust generation

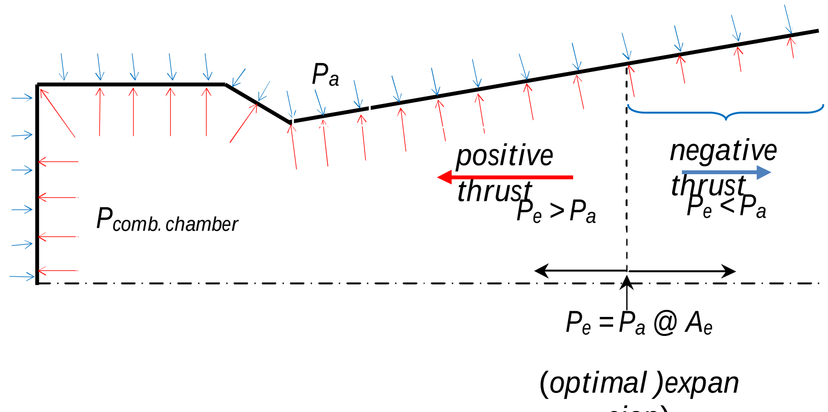

Figure 3. Schematic showing optimal expansion in a CD nozzle.

Exhaust gas velocity (U_e) in terms of combustion chamber properties

Ideal analysis: one dimensional, steady state, isentropic flow. The gas is assumed to be a perfect gas with constant properties

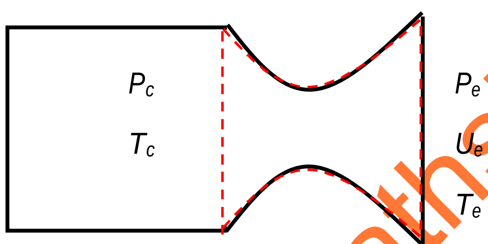

Figure 4 A schematic showing control volume to relate exhaust velocity (U_e) to thermodynamic properties in the combustion chamber.

Conservation of energy

$$\frac{\partial}{\partial t}\iiint_V \left(e_0 + \frac{U^2}{2}\right)\rho dV + \oiint_S \left(h + \frac{U^2}{2}\right)\rho \vec{U}\cdot d\vec{S} = \delta\dot{Q} - \delta\dot{W}'$$

$$h_c + \frac{V_{c1}^2}{2} = h_e + \frac{U_e^2}{2}$$

$$2C_p(T_c - T_e) = U_e^2 - V_c^2 \qquad \text{----(1)}$$

$$C_p - C_v = R, \quad \frac{C_p}{C_v} = \gamma \quad \text{and} \quad R = \frac{R_u}{M_W}$$

eliminating C_v we get $C_p = R\frac{\gamma}{\gamma - 1} = \frac{R_u}{M_W} \times \frac{\gamma}{\gamma - 1}$

for *isentropic flow* the pressure temperature relationship follows

$$\frac{T_e}{T_c}=\left(\frac{p_e}{p_c}\right)^{\frac{r-1}{r}}, \text{ or } \qquad T_c-T_e=T_c\left[1-\left(\frac{p_e}{p_c}\right)^{\frac{r-1}{r}}\right] \qquad \text{----(2)}$$

Substituting for (T_c-T_e) in (1)

$$U_e=\left(\frac{T_c}{M_W}\times\frac{2R_u\gamma}{\gamma-1}\left[1-\left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]+V_c^2\right)^{1/2} \qquad (3)$$

Now

V_c = gas flow velocity in combustion chamber is very small and hence neglected

p_e = outside atmospheric pressure $<<p_c \quad \Rightarrow \frac{p_e}{p_c}$ is very small.

$$\therefore U_e = \text{const}\times\sqrt{\frac{T_c}{M_W}} \qquad (4)$$

Note: Exhaust velocity, U_e and thus the specific impulse, I_{sp} is directly proportional to combustion temperature, T_c and inversely proportional to the molecular weight, M_W of combustion gases $\Rightarrow$ a criterion for selecting propellant

Specific Impulse (I_{sp}) of representative space propulsion system

Engine type	Working fluid	Specific impulse
Chemical (liquid)		
Monopropellant	Hydrogen peroxide-hydrazine	110–140 220–245
Bipropellant	O_2-H_2	440–480
	O_2-hydrocarbon	340–380
	N_2O_4-Monomethyl hydrazine	300–340
Chemical (solid)	Fuel and oxidizer	260–300
Nuclear	H_2	600–1000
Solar heating	H_2	400–800
Arc jet	H_2	400–2000
Cold gas	N_2	50–60
Ion	Cesium	5000–25,000

2. ROCKET DYNAMICS

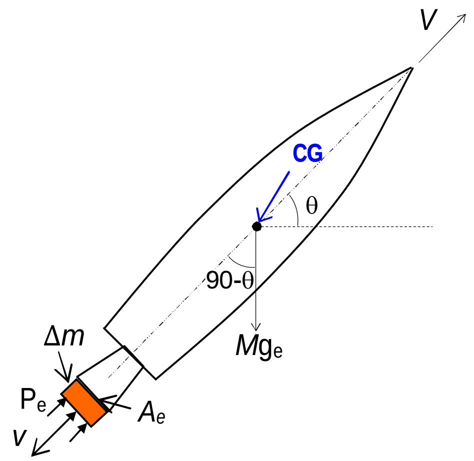

Figure 1

Consider an accelerating rocket vehicle (figure. 1). Let '*M*' be the mass of the vehicle at time '*t*' and '*V*' its velocity. Over a short time interval 'Δt' the rocket ejects mass 'Δm' with velocity '*v*' and the vehicle acquires velocity '$V+\Delta V$'.

The change in momentum of rocket: $(M-\Delta m)(V+\Delta V)-(M-\Delta m)V$

The change in momentum of mass Δm: $\Delta m(v)-\Delta m(V)$

The change in momentum of the system: $M\Delta V-\Delta m(V-v)$

The net external forces on the system: $A_e(p_e-p_a)-D-Mg_e\sin\theta$

$$M\Delta V-\Delta m(V-v)=\left[A_e(p_e-p_a)-D-Mg_e\sin\theta\right]\Delta t$$

$$M\underset{\Delta t\to 0}{Lt}\frac{\Delta V}{\Delta t}=\left[\underset{\Delta t\to 0}{Lt}\frac{\Delta m}{\Delta t}(V-v)+A_e(p_e-p_a)-D-Mg_e\sin\theta\right]$$

$M\dfrac{dV}{dt}=\left[\dot{m}(V-v)+A_e(p_e-p_a)-D-Mg_e\sin\theta\right]$ where $\underset{\Delta t\to 0}{Lt}\dfrac{\Delta m}{\Delta t}=\dot{m}$

Now $T=\dot{m}(V-v)+A_e(p_e-p_a)=\dot{m}U_e+A_e(p_e-p_a)=\dot{m}U_{eq}$

Rearranging we get general **Equation of Motion** for a rocket vehicle

$$\textit{Vehicle accleation},\quad a=\frac{dV}{dt}=\frac{T}{M}-\frac{D}{M}-g_e\sin\theta$$

Typically for a launch vehicle, launched from the earth, the thrust to weight ratio, $\left(\dfrac{T}{Mg_e}\right)$, is between 1.5 to 2 to ensure reasonable acceleration.

Case I (Neglecting the atmospheric drag and the effect of gravity)

$$\frac{dV}{dt}=\frac{T}{M}=\frac{\dot{m}\left[(V-v)+A_e\left(p_e-p_a\right)/\dot{m}\right]}{M}=-\frac{U_{eq}}{M}\frac{dM}{dt}$$

$$dV=-U_{eq}\frac{dM}{M}$$

Integrating over time '0' to '*t*'

$$\Delta V=\int_{V_0}^{V_F}dV=-U_{eq}\int_{M_0}^{M}\frac{dM}{M}=U_{eq}\ln\frac{M_0}{M}$$

We get **Rocket Equation** (*Konstantin Tsiolkovsky's* Rocket Equation)

$$\Delta V_{ideal}=U_{eq}\ell n\frac{M_0}{M}=I_{sp}g_e\ell n\frac{M_0}{M} \quad \text{--------------- (1)}$$

Here: M_0 = initial mass of the vehicle,

M = instantaneous mass of the vehicle a time '*t*'.

Total velocity increment over the acceleration phase lasting for time 't_b'

$$\Delta V_{\max}=I_{sp}g_e\ell n\frac{M_0}{M_B} \quad \text{-------------- (2)}$$

where M_B is the mass of the vehicle at burnout

Plotting $\Delta V_{\max}$ vs (M_o/M) for different values of I_{sp} using eqn. (2), the following points can be observed from figure 2.

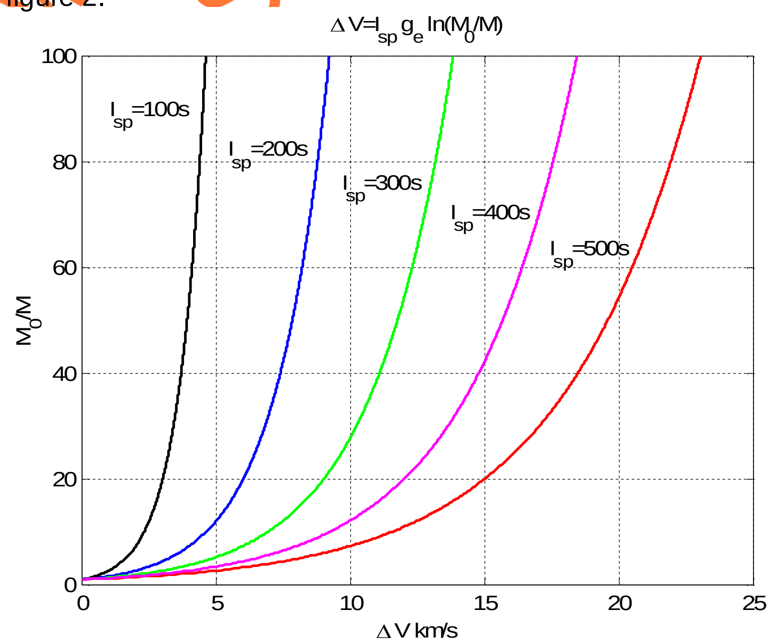

Figure 2