(b) The time period is

$$T = \frac{2\pi a}{v} = \frac{2\pi \times 7000 \times 10^3}{7.6 \times 10^3}$$

$$T = 5.8 \times 10^3 \text{ sec}$$

=1hr, 37min, 8.5sec.

Kepler's laws:-

Kepler's listed the laws of planetary motion.

Kepler's first major conclusions are as follows:

Ist Law: - A satellite describes an elliptical path around its centre of attraction.

IInd Law: - In equal times, the areas swept out by the radius vector of a satellite are the same.

$$\frac{dA}{dt} = \text{constant}$$

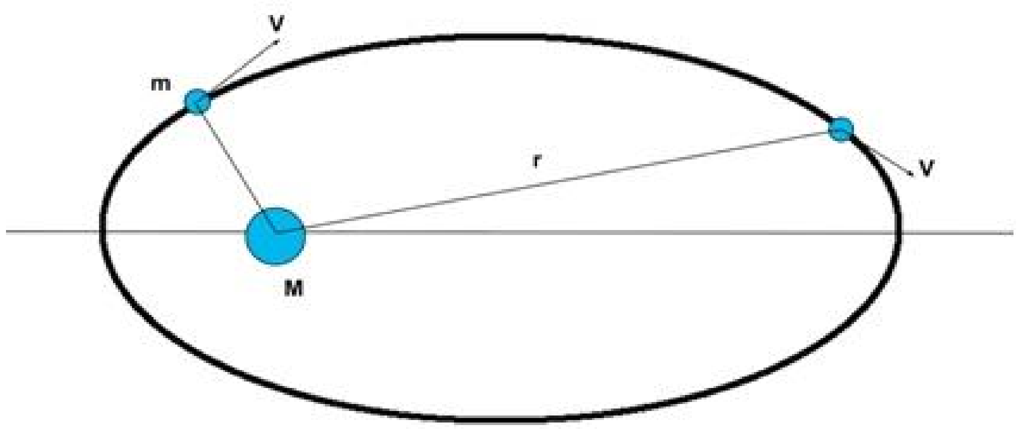
IIIrd Law: -The Square of the time period of a planet is proportional to the cube of the semi major axis.

$$T^2 \alpha\, a^3$$

Characteristics of satellites in elliptical orbit:-

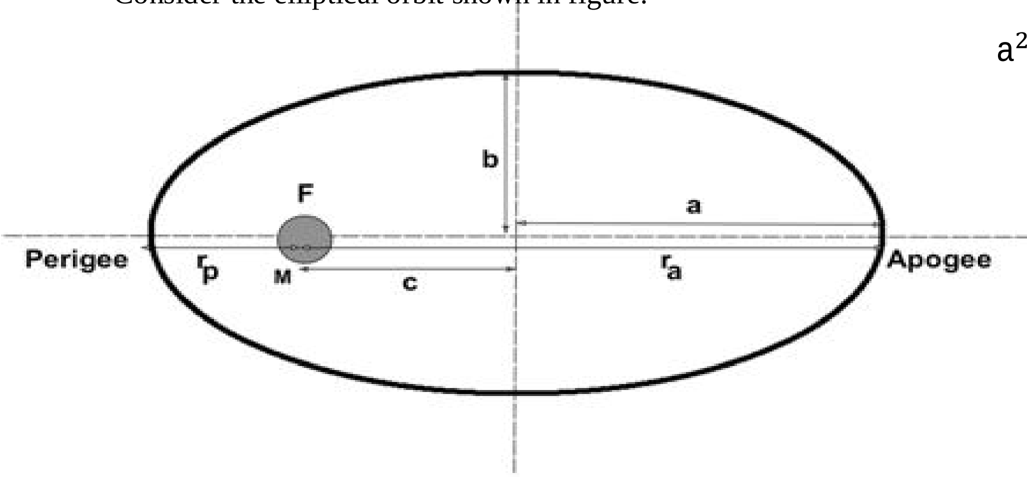
From IInd Law, conclusion follows as:

Here, the elliptical orbit of a small mass is shown about a large mass M. In order for equal areas to be swept out in equal times, the satellite must have a larger velocity when it is near M and a smaller velocity when it is far away. This is characteristic of all satellite motion.



Derivation of Kepler's III[rd] Law:-

Consider the elliptical orbit shown in figure.



$$a^2 = b^2 + c^2$$

$$e = \frac{c}{a}$$

The point of closest approach where r is minimum, is defined as the perigee, the point farthest away, where r is maximum is defined as the apogee.

The mass M is at the focus of ellipse.

a – semi major axis

b – semi minor axis.

Let us assume for simplicity that the phase angle 'c' of the orbit is zero.

Thus from vehicle trajectory

$$r = \frac{P}{1 + e\cos\theta}$$

$$r_a = r_{max} = \frac{h^2/k^2}{1-e} = \frac{P}{1-e} \quad \text{(Therefore } \theta = \text{pi)}$$

$$r_p = r_{min} = \frac{h^2/k^2}{1+e} = \frac{P}{1+e} \quad \text{(Therefore } \theta = 0)$$

$$a = \frac{r_{max} + r_{min}}{2} = \frac{r_a + r_p}{2}$$

$$= \frac{1}{2} h^2/k^2 \left[\frac{1}{1-e} + \frac{1}{1+e}\right] = \frac{h^2/k^2}{1-e^2}$$

$$a = \frac{1}{2}(r_a + r_p) = \frac{h^2/k^2}{1-e^2}$$

from analytical geometry, Eccentricity, $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{c}{a}$

Solving for b;

$$b = a\sqrt{1 - e^2}$$

Area of ellipse, $A = \pi ab$

$$A = \pi a [a\sqrt{1-e^2}]$$

$$\boxed{A = \pi a^2(1-e^2)^{1/2}} \qquad \rightarrow \text{(ii)}$$

Consider the elemental area (dA) swept by an satellite in time dt.

$$dA = \frac{1}{2} rds$$

$$dh = rd\,\theta$$

$$dA = \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} r^2 w = \text{constant}$$

$$\frac{dA}{dt} = \frac{1}{2} h \Rightarrow \quad w \,\alpha\, \frac{1}{r^2} \text{ (Therefore } h = r^2\dot{\theta})$$

Therefore $\boxed{dA = \frac{1}{2} r^2 \dot{\theta}\, dt = \frac{1}{2} h\, dt}$

The area swept out by the radius vector is the whole area of the ellipse A. The time taken by the satellite in executing the complete orbit is defined as the period and is denoted by T. Integrating equation (iii) around the complete orbit, we get

$$\int_o^A dA = \int_o^T \frac{1}{2} h\, dt$$

$$A = \frac{1}{2} h\, T$$

From (ii) $\frac{1}{2} h\, T = \pi a^2 \sqrt{1-e^2}$

From (i) $a = \frac{h^2/k^2}{1-e^2} \Rightarrow h = \sqrt{a}\; k\sqrt{1-e^2}$

Substituting in above equation, we get

$$\frac{1}{2} h\, T = \frac{1}{2} h = \sqrt{a}\; k\sqrt{1-e^2}\, T = \pi a^2(1-e^2)^{1/2}$$

$$\frac{1}{4} T^2 a\, k^2 = \pi^2 a^4$$

$$T^2 = \frac{4\pi^2}{k^2} a^3$$

$$(\text{Since } k^2 = GM)$$

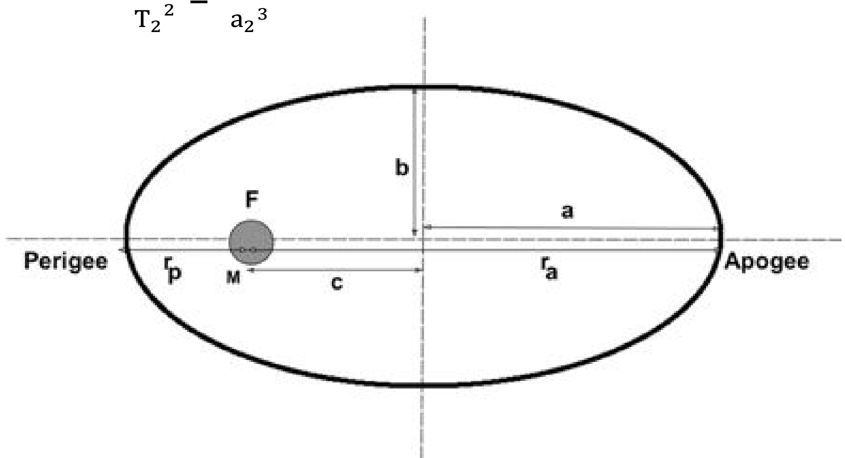
$$T^2 = \frac{4\pi^2}{GM} a^3$$

$$\text{Therefore } T^2 = (\text{constant})a^3$$

If we have two satellites in orbit about the same planet, with values of T_1, a_1 and T_2, a_2 respectively. Then Kepler's 3rd law can be written asfollows;

$$\frac{T_1^{\,2}}{T_2^{\,2}} = \frac{a_1^{\,3}}{a_2^{\,3}}$$

For Elliptical orbit:-



$r_p = a - c$

$r_a = a + c$

$r_p + r_a = 2a$

$r_a - r_p = 2c$

Eccentricity $e = \frac{c}{a} = \frac{r_a - r_p}{r_a + r_p}$

$$r_a = a\,(1 + e); \qquad r_p = a\,(1 - e)$$

From orbit equation with c = 0

$$r = \frac{P}{1 + e\cos\theta}$$

$$r_p = \frac{P}{1+e}; \qquad r_a = \frac{P}{1-e}$$

Velocity at any point in orbit at radius r.

$$V^2 = GM_e\left[\frac{2}{r} - \frac{1}{a}\right]$$

$$V = \sqrt{GM_e\left[\frac{2}{r} - \frac{1}{a}\right]}$$

⇨ Velocity at periapsis; $r_p = a\,(1 - e)$

$$V_p = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

⇨ Velocity at apoapris, $r_a = a\,(1 + e)$

$$V_a = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

⇨ Velocity needed to escape the orbit is given by $E_T = 0 = \frac{1}{2}V^2 - \frac{\mu}{r}$

$$\text{Therefore } V_e = \sqrt{\frac{2GM}{r}}$$

⇨ If ΔV escape, excess velocity required to escape orbit from a particular point on the orbit

$$\Delta V\,\text{escape} = \sqrt{\frac{2GM}{r}} - \sqrt{GM\left[\frac{2}{r} - \frac{1}{a}\right]}$$

@ Periapsis

$$\Delta V\,\text{escape, P} = \sqrt{\frac{2GM}{a\,(1-e)}} - \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

and @ Apoapsis

$$\Delta V\,\text{escape, a} = \sqrt{\frac{2GM}{a\,(1+e)}} - \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

Example (1):- Find the distance of a point from the earth's centre where the resultant gravitational field due to the earth and the moon is zero. Mass of the moon is 1.4×10^{22} kg. The distance between the earth and the moon is 4×10^{5} km.

(a) 1.6×10^{5} km

(b) 3.01×10^{5} km

(c) 3.60×10^{5} km

(d) 3.2×10^{5} km

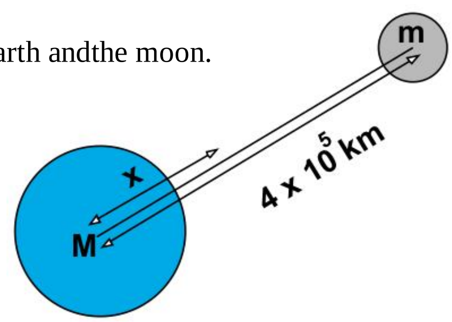
Sol:- the pt must be on the line joining the centre of the earth andthe moon.

Let the distance of the point from earth = x

The distance of the point from moon = $(4 \times 10^{5} - x)$

Gravitational field due to the earth, $E_1 = \frac{GM_e}{x^2}$

Gravitational field due to the moon, $E_2 = \frac{GM_m}{(4 \times 10^5 - x)^2}$



These fields are in opposite direction for the resultant field to be zero.

$$E_1 = E_2$$

$$\frac{GM_e}{x^2} = \frac{GM_m}{(4 \times 10^5 - x)^2}$$

$$\frac{6 \times 10^{24}}{x^2} = \frac{7.4 \times 10^{22}}{(4 \times 10^5 - x)^2}$$

$$\frac{x}{4 \times 10^5 - x} = \sqrt{81} = 9$$

$$x = 3.6 \times 10^5 \text{ km}$$